

Phase structure and asymptotic zero densities of orthogonal polynomials in the cubic model

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Abstract

We apply the method we have described in a previous paper (2013) to determine the phase structure of asymptotic zero densities of the standard cubic model of non-hermitian orthogonal polynomials. We provide a complete description of the two phases: the one cut phase and the two cut phase, and analyze the phase transition processes of the types: splitting of a cut, birth and death of a cut.

Keywords: standard cubic model, phase transitions, orthogonal polynomials

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1. Introduction

Orthogonal polynomials in which the weight can be a complex function and the integration path can be a general curve in the complex plane (nonhermitian orthogonal polynomials) first appeared in the mathematical literature as denominators of Padé and other types or rational approximants [23, 24, 25, 26]. The theory quickly developed and found applications into such fields as the Riemann-Hilbert approach to strong asymptotics, random matrix theory [10, 4, 5, 6, 3, 2] and in the study of dualities between supersymmetric gauge theories and string models [18, 7, 11, 12, 15]. This

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paper is devoted to a detailed analysis of the phase structure and phase transitions of the asymptotic (in the limit $n \rightarrow \infty$) zero density of monic orthogonal polynomials $P_n(z) = z^n + \dots$

$$\int_{\Gamma} P_n(z) z^k e^{-nW(z)} dz = 0, \quad k = 0, \dots, n-1, \quad (1)$$

where

$$W(z) = \frac{z^3}{3} - tz, \quad (2)$$

and t is an arbitrary complex number. The curve Γ is an infinite path connecting any two of the three different sectors shaded in figure 1 on which $\text{Re}(z^3) > 0$, so that the integral in (1) is convergent and nonzero. Since the three possible choices are trivially related, without loss of generality we choose Γ as the curve connecting the two sectors with $\text{Re}(z) < 0$.

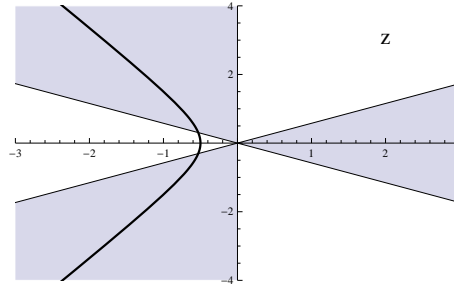


Figure 1: The three regions of the complex z plane with $\text{Re}(z^3) > 0$ and the curve Γ .

The particular case $t = 0$ of this model has been rigorously studied by Deaño, Huybrechs and Kuijlaars [9], and recent results for $t \in \mathbb{R}$ have been communicated by Lejon [16]. The phase structure of the corresponding random matrix model has been studied by David [8] and Mariño [18].

To perform our analysis we follow the method described in [1], which is applicable to study orthogonal polynomials with respect to general potentials

$$W(z) = \sum_{k=1}^N t_k z^k, \quad (3)$$

and appropriate choices of Γ . This method is based on the existence of a unique normalized equilibrium charge density that minimizes the electrostatic energy (among all normalized charge densities supported on the curve

Γ) in the presence of the external electrostatic potential $V(z) = \operatorname{Re} W(z)$ [21]. The method in [1] also relies heavily on the concepts of S -property and S -curve of Stahl [23, 24, 25, 26] and of Gonchar and Rakhmanov [14, 19, 20], and on the fundamental result of Gonchar and Rakhmanov [14] asserting that if Γ is an S -curve, then the asymptotic zero density of $P_n(z)$ exists and is given by the equilibrium charge density of the associated electrostatic model. Note that the integral (1) is invariant under deformations of the curve Γ into curves in the same homology class and connecting the same two convergence sectors at infinity. Recent results of Rakhmanov [20] and of Kuijlaars and Silva [17] show that for any family of orthogonal polynomials of the form (1) there always exists a deformation of Γ into an appropriate S -curve. We will make extensive use of certain algebraic curves (called *spectral curves*) which have the form

$$y^2 = W'(z)^2 + f(z), \quad (4)$$

where $f(z)$ is a polynomial of degree $\deg f = \deg W - 2$. More concretely, the main parameters that determine the S -curves and the associated equilibrium densities are the branch points of $y(z)$, which turn out to be the endpoints of the (in general, several disjoint) arcs that support the equilibrium density. The corresponding cuts are characterized as Stokes lines of the polynomial $y(z)^2$ or, equivalently, as trajectories of the quadratic differential $y(z)^2(dz)^2$. At this point numerical analysis will be necessary not only to solve the equations for the endpoints but also to analyze the existence of cuts satisfying the S -property.

The paper is organized as follows. In section 2 we recall the main ingredients of the method presented in [1], namely, the relation between the electrostatic problem and the asymptotic zero density of the orthogonal polynomials, the S -property, and the notion of spectral curve. In section 3 we determine the region of the complex t -plane in which the asymptotic density of zeros of the orthogonal polynomials is supported on a single cut (the one-cut phase of the cubic model (2)), and for each t in this region we calculate the corresponding S -curve and zero density. Since the number of disjoint arcs on which the asymptotic density of zeros is supported is less than or equal to $\deg W - 1$, the complement of the one-cut phase in the complex t -plane must be the region wherein the asymptotic density of zeros is supported on two disjoint arcs (the two-cut phase); we study this case in section 4. Section 5 is dedicated to the study of phase transitions. We characterize critical processes of splitting, birth and death at a distance of cuts. The consistency

of our results is checked in section 6 by superimposing the cuts and the zeros of the corresponding orthogonal polynomials $P_n(z)$ with degree $n = 36$.

2. Zero densities of orthogonal polynomials and spectral curves

According to the general theory of logarithmic potentials with external fields [21], given an analytic curve Γ in the complex plane and a real-valued external potential $V(z)$, there exists a unique charge density $\rho(z)$ that minimizes the total electrostatic energy

$$\mathcal{E}[\rho] = \int_{\Gamma} |dz| \rho(z) V(z) - \int_{\Gamma} |dz| \int_{\Gamma} |dz'| \log |z - z'| \rho(z) \rho(z') \quad (5)$$

among all positive densities supported on Γ such that

$$\int_{\Gamma} |dz| \rho(z) = 1. \quad (6)$$

This density $\rho(z)$ is called the *equilibrium density*, and its support γ is a finite union of disjoint analytic arcs γ_i (cuts) contained in Γ :

$$\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_s \subset \Gamma. \quad (7)$$

In terms of the total electrostatic potential

$$U(z) = \frac{\delta \mathcal{E}}{\delta \rho(z)} = V(z) - 2 \int_{\Gamma} |dz'| \rho(z') \log |z - z'|, \quad (8)$$

the equilibrium density is characterized by the existence of a real constant l such that

$$U(z) = l, \quad z \in \gamma, \quad (9)$$

$$U(z) \geq l, \quad z \in \Gamma - \gamma. \quad (10)$$

The property that relates this minimization problem to the asymptotic zero density of orthogonal polynomials is called the *S-property*, and was singled out by Stahl [23, 24, 25, 26], elaborated by Gonchar and Rakhmanov [14, 13], and more recently extended by Martínez-Finkelshtein and Rakhmanov [19].

A curve Γ is said to be an *S-curve* with respect to the external field $V(z)$ if for every interior point z of the support γ of the equilibrium density the total potential (8) satisfies

$$\frac{\partial U(z)}{\partial n_+} = \frac{\partial U(z)}{\partial n_-}, \quad (11)$$

where n_{\pm} denote the two normal vectors to γ at z pointing in the opposite directions. In this case it is said that γ satisfies the S -property. The condition (11) means that the electric fields at each side are opposite, $\mathbf{E}_+ = -\mathbf{E}_-$.

The following Theorem (see [14], Section 3) states the close relation between the asymptotic zero densities of orthogonal polynomials and the equilibrium densities on S -curves:

Theorem 1. *Let $\{P_n(z)\}_{n \geq 1}$ be a family of orthogonal polynomials on a curve Γ with respect to an exponential weight $\exp(-nW(z))$. If Γ is an S -curve with respect to the external potential $V(z) = \operatorname{Re} W(z)$, then the equilibrium density on Γ is the weak limit as $n \rightarrow \infty$ of the zero density of $P_n(z)$.*

It often occurs in the applications that the orthogonal polynomials $P_n(z)$ are initially defined on a curve Γ that connects two different convergence sectors at infinity of the integral (1), but which is not an S -curve. This problem raises the question of the existence of an S -curve in the same homology class of Γ connecting the same pair of convergence sectors at infinity (and therefore defining the same family of orthogonal polynomials). This question has been recently solved in the affirmative by Rakhmanov [20] and Kuijlaars and Silva [17]. Note also that although this S -curve is not unique, the associated equilibrium density is certainly unique.

To obtain an alternative characterization of the S -property we introduce complex counterpart of the electrostatic potential (8). Thus, we define the *complex electrostatic potential* $\mathcal{U}(z)$ as

$$\mathcal{U}(z) = W(z) - (g(z_+) + g(z_-)), \quad (12)$$

where $g(z)$ is the analytic function in $\mathbb{C} \setminus \Gamma$ given by

$$g(z) = \int_{\gamma} |dz'| \rho(z') \log(z - z'). \quad (13)$$

Here, $g(z_+)$ and $g(z_-)$ denote the limits of the function $g(z')$ as z' tends to z from the left and from the right of the oriented curve Γ respectively, and the logarithm branches have to be chosen so that $\log(z_+ - z') + \log(z_- - z') = \log(z'_+ - z) + \log(z'_- - z)$ for z and z' in Γ . It is clear that

$$\operatorname{Re} \mathcal{U}(z) = U(z), \quad (14)$$

and therefore the equilibrium condition (9) can be rewritten as

$$\operatorname{Re} \mathcal{U}(z) = l, \quad z \in \gamma. \quad (15)$$

Furthermore, it follows from the Cauchy-Riemann equations that the S -property (11) is verified if and only if the imaginary part of $\mathcal{U}(z)$ is constant on each arc γ_j of γ (usually stated as “locally constant on γ ”) [9, 19]. From (15) it is obtained the following characterization of S -curves in terms of the complex potential:

Theorem 2. *Γ is an S -curve if and only if the complex potential $\mathcal{U}(z)$ is locally constant on γ*

$$\mathcal{U}(z) = L_j, \quad z \in \gamma_j, \quad j = 1, \dots, s, \quad (16)$$

and the constants L_j have the same real part

$$\operatorname{Re} L_1 = \dots = \operatorname{Re} L_s = l. \quad (17)$$

Moreover, condition (16) can be rewritten (see [1] for details) in a form especially suited for practical applications in terms of a new function $y(z)$ defined by

$$y(z) = W'(z) - 2g'(z) = W'(z) - 2 \int_{\gamma} |dz'| \frac{\rho(z')}{z - z'}, \quad z \in \mathbb{C} \setminus \Gamma. \quad (18)$$

Proposition 1. *The complex potential $\mathcal{U}(z)$ is locally constant on γ if and only if the square of $y(z)$ is a polynomial of the form*

$$y(z)^2 = W'(z)^2 + f(z), \quad (19)$$

where $f(z)$ is a polynomial of degree $\deg f = \deg W - 2$.

From (18) it is clear that

$$y(z) = W'(z) - \frac{2}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \quad \text{as } z \rightarrow \infty. \quad (20)$$

On the other hand, equation (19) defines an algebraic curve referred to as a *spectral curve*, which determines the equilibrium charge density via (18),

$$\rho(z)|dz| = y(z_+) \frac{dz}{2\pi i} = -y(z_-) \frac{dz}{2\pi i}, \quad z \in \gamma, \quad (21)$$

where γ has the orientation inherited by the orientation of Γ .

As an immediate consequence of (19) and (21) we see that the number of cuts for a fixed $W(z)$ is at most $\deg W - 1$. In particular, for the cubic model (2) the zeros of the orthogonal polynomials in the limit $n \rightarrow \infty$ can be supported only in one or in two cuts. We consider these two cases separately in the next two sections.

3. The one cut case for the cubic model

To determine the branch $y(z)$ for the cubic model (2) that satisfies (18) in the one-cut case we write $y(z)$ in the form

$$y(z) = h(z)\omega(z), \quad (22)$$

with

$$\omega(z) = \sqrt{(z-a)(z-b)}, \quad \omega(z) \sim z \quad \text{as } z \rightarrow \infty. \quad (23)$$

Here a and b are the endpoints of the cut and $h(z)$ is a linear polynomial. From (20) and (22) we have that

$$h(z) = \left(\frac{W'(z)}{\omega(z)} \right)_{\oplus} = z + \frac{a+b}{2}, \quad (24)$$

where \oplus stands for the sum of the nonnegative powers of the Laurent series at infinity. On the other hand, from (19) we also obtain that

$$y(z)^2 = (z^2 - t)^2 - 4z + b_0, \quad (25)$$

with b_0 a certain complex number. By comparing (22)–(24) with (25) we find a system of equation for the endpoints a and b , which is more conveniently written in terms of

$$\beta = \frac{a+b}{2}, \quad \delta = \frac{b-a}{2}, \quad (26)$$

and reads

$$2\beta^2 + \delta^2 = 2t, \quad (27)$$

$$\beta\delta^2 = 2. \quad (28)$$

Therefore β satisfies the cubic equation

$$\beta^3 - t\beta + 1 = 0, \quad (29)$$

and δ is determined by

$$\delta = \sqrt{2t - 2\beta^2}. \quad (30)$$

Thus, for each complex t , the cubic equation (29) has three solutions that can be written as

$$\beta_k(t) = -\frac{t}{3\Delta_k} - \Delta_k, \quad (k = 0, 1, 2) \quad (31)$$

where

$$\Delta_k = e^{i2\pi k/3} \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{t}{3}\right)^3}}, \quad (32)$$

and where the roots take their respective principal values. There are three finite branch points

$$t^{(k)} = \frac{3}{2^{2/3}} e^{i2\pi k/3}, \quad (k = 0, 1, 2) \quad (33)$$

at which $\beta_1(t^{(0)}) = \beta_2(t^{(0)})$, $\beta_0(t^{(1)}) = \beta_1(t^{(1)})$, and $\beta_0(t^{(2)}) = \beta_2(t^{(2)})$ respectively, and each solution $\beta_k(t)$ is discontinuous on two rays in the complex t -plane: the function $\beta_0(t)$ is discontinuous on the rays with negative $\text{Re}(t)$, the function $\beta_1(t)$ is discontinuous on the ray on the real axis and on the ray with $\text{Arg}(t) = 2\pi/3$, and $\beta_2(t)$ is discontinuous on the ray on the real axis and on the ray with $\text{Arg}(t) = -2\pi/3$. These branch points and branch cuts are illustrated in figure 2.

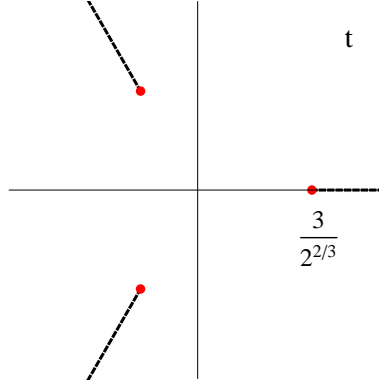


Figure 2: Branch points and branch cuts for the functions $\beta_k(t)$.

Using (26) and (30), each function $\beta_k(t)$ yields a possible pair of endpoints $a(t)$ and $b(t)$

$$a_k(t) = \beta_k(t) - \sqrt{2(t - \beta_k(t))}, \quad b_k(t) = \beta_k(t) + \sqrt{2(t - \beta_k(t))}. \quad (34)$$

We need to find out which of these three candidates, if any, are the actual endpoints of the arc on which the zeros of the orthogonal polynomials concentrate.

To this aim we introduce the functions

$$G(z) = \int_a^z y(z'_+) dz'. \quad (35)$$

In [1] we show that

$$\operatorname{Re} G(z) = U(z) - l, \quad z \in \mathbb{C} \setminus \gamma. \quad (36)$$

Consequently, the equilibrium condition (9) can be written in terms of $G(z)$ as

$$\operatorname{Re} G(z) = 0, \quad z \in \gamma. \quad (37)$$

Given a root z_0 of $y(z)^2$ with multiplicity m , there are $m + 2$ maximal connected components (excluding any zeros of $y(z)^2$) of the level curve

$$\operatorname{Re} \int_{z_0}^z y(z'_+) dz' = 0, \quad (38)$$

which stem from z_0 [22]. These maximal components are called the Stokes lines outgoing from z_0 associated to the polynomial $y(z)^2$. Stokes lines for a polynomial cannot make loops and end necessarily either at a different zero of $y(z)$ (lines of *short* type) or at infinity (lines of *leg* type).

Therefore, for (a_k, b_k) to be the endpoints of the support, a Stokes line starting at a_k must reach b_k . However, this is not a sufficient condition, because (10) must also to be satisfied, i.e., we also require the existence of an infinite line Γ' containing the Stokes line from a_k to b_k , connecting the same two sectors of infinity as Γ , and such that

$$\operatorname{Re} G(z) > 0, \quad z \in \Gamma' \setminus \gamma. \quad (39)$$

The function $G(z)$ also allows us to characterize the condition for a Stokes line starting from a to be critical. Indeed, the derivatives of $\operatorname{Re} G$ with respect to the cartesian coordinates are

$$\frac{\partial}{\partial x} \operatorname{Re} G(z) = \operatorname{Re} y(z), \quad \frac{\partial}{\partial y} \operatorname{Re} G(z) = -\operatorname{Im} y(z), \quad (40)$$

so that a critical point $z_c \neq a, b$ appears in a Stokes line if

$$\operatorname{Re} G(z_c) = 0, \quad y(z_c) = 0. \quad (41)$$

This situation arises in particular at phase transition of equilibrium densities in which the number of cuts changes.

Coming back to the cubic model, equations (22)–(24) and (26) lead to

$$y(z) = (z + \beta_k) \sqrt{(z - \beta_k)^2 - \delta_k^2}, \quad (42)$$

and using (35) we find that the G_k function corresponding to the branch $\beta_k(t)$ (with initial integration point $a_k(t)$ defined in (34)) is

$$\begin{aligned} G_k(z) = & \frac{1}{3} \sqrt{(z - \beta_k)^2 - \delta_k^2} \left(z^2 + \beta_k z + \beta_k^2 - 3t + \frac{\delta_k^2}{2} \right) \\ & - \log \left(\frac{\beta_k - z - \sqrt{(z - \beta_k)^2 - \delta_k^2}}{\delta_k} \right)^2. \end{aligned} \quad (43)$$

The only possible critical point is $z_c = -\beta_k$, so that the critical condition (41) can be written as

$$\operatorname{Re} G_k(-\beta_k(t)) = 0, \quad (44)$$

where

$$G_k(-\beta_k) = -\frac{1}{3} \sqrt{4\beta_k^2 - \delta_k^2} (2\beta_k^2 + \delta_k^2) - \log \left(\frac{2\beta_k - \sqrt{4\beta_k^2 - \delta_k^2}}{\delta_k} \right)^2. \quad (45)$$

In figure 3 we show the curves in the complex t -plane determined by the solutions of (44). The orange arcs correspond to $k = 0$, the blue arcs to $k = 1$ and the green arcs to $k = 2$. In addition each region has been identified with a number that will be used in our forthcoming discussion of the phase structure.

We come back to the problem of deciding which one of the pairs of possible endpoints (34), determines the cut. By continuity, in each region labelled in figure 3 the solution is given for the same value of k , so that we have to consider the problem separately in each of these regions in the t -complex plane. We discuss next four of these sectors.

Figure 4 illustrates the situation for t in region 1. The left plot corresponds to β_0 , the center plot to β_1 and the right plot to β_2 . In the three cases there exists a short Stokes line connecting the corresponding a_k and b_k . However, only for β_0 , the short line can be continued into the set $\operatorname{Re}(G(z)) > 0$ to a curve homologue to Γ . Thus, region 1 is in the one-cut sector and the S -curve is provided by β_0 .

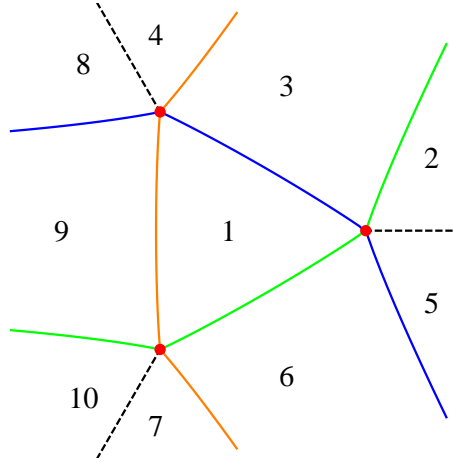


Figure 3: The solid lines represent the solutions of (44) for $k = 0$ (orange), $k = 1$ (blue), and for $k = 2$ (green), superimposed to the branch points and branch cuts of figure 2.

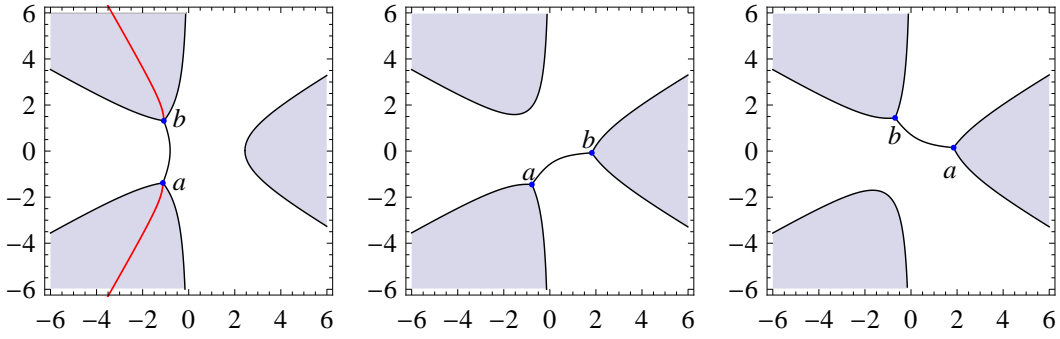


Figure 4: Stokes lines and S -curve for t in region 1. The left plot corresponds to β_0 , the center plot to β_1 and the right plot to β_2 . Regions with $\text{Re}(G(z)) > 0$ are shaded.

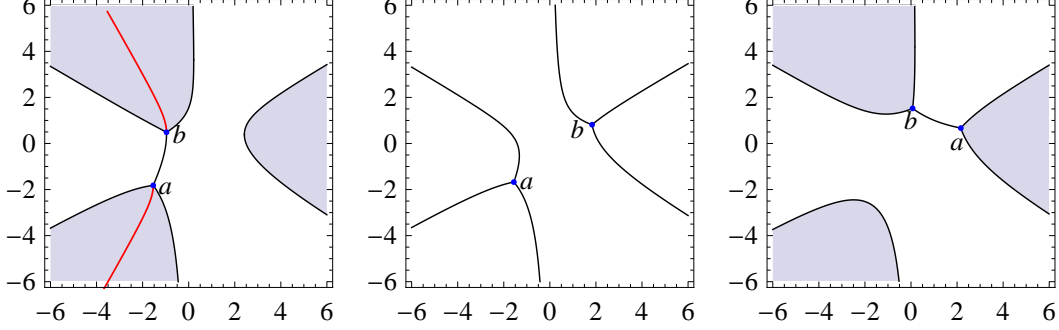


Figure 5: Stokes lines and S -curve for t in region 3. The left plot corresponds to β_0 , the center plot to β_1 and the right plot to β_2 . Shaded regions in the first and third plot correspond to $\text{Re}(G(z)) > 0$.

We consider now t in region 3. The Stokes graphs associated to the three candidates to endpoints are shown in figure 5. Note that in this region there is not a Stokes line connecting a and b for β_1 . There exist Stokes lines with this property for β_0 and β_2 , but only in the first case the Stokes line can be continued into the set $\text{Re}(G(z)) > 0$ to a curve homologue to Γ . We have in consequence that region 3 is in the one cut sector and the S -curve is again provided by β_0 .

The next example we discuss is t in region 8. Figure 6 shows that there is not a Stokes line connecting the a and b endpoints associated to β_0 , but that such a Stokes line exists both for β_1 and β_2 . Only in the first case the Stokes line can be continued into the set $\text{Re}(G(z)) > 0$ to a curve homologue to Γ . Thus, region 8 is in the one cut sector but the S -curve is now provided by β_1 .

Finally we consider t in region 9, and plot the Stokes lines in figure 7. As in the previous case, only for β_1 and β_2 there exists a Stokes line connecting a and b , but differently from all the previous examples, in neither case the Stokes line cannot be prolonged into $\text{Re}(G(z)) > 0$ to a curve homologue to Γ . We have seen then that region 9 is not in the one cut sector, so it has to belong to the two cut phase.

Proceeding in the same way with regions 2, 4, 5, 6, 7 and 10, we find that all this regions are in the one cut phase, and the solution is provided by β_0 except in region 10, where the cut is associated to β_2 . We summarize these

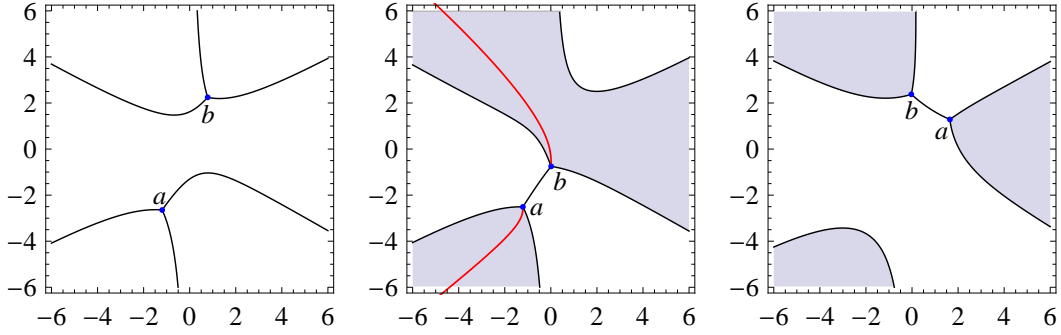


Figure 6: Stokes lines and S curve for t in region 8. The left plot corresponds to β_0 , the center plot to β_1 and the right plot to β_2 . Shaded regions in the second and third plots correspond to $\text{Re}(G(z)) > 0$.

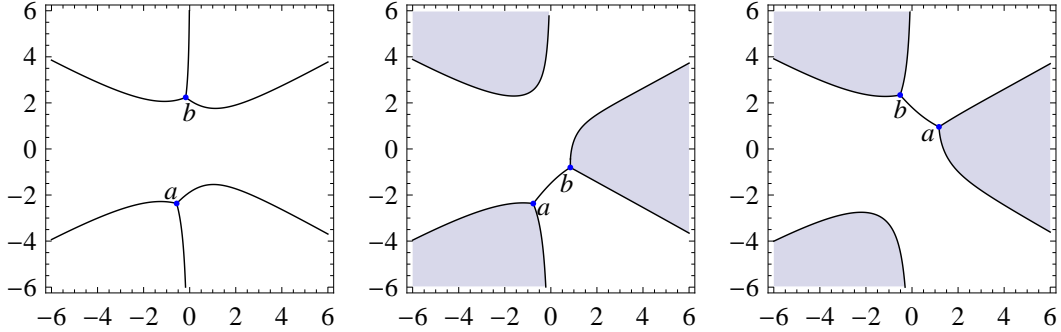


Figure 7: Stokes lines for t in region 9. The left plot corresponds to β_0 , the center plot to β_1 and the right plot to β_2 . Shaded regions in the second and third plots correspond to $\text{Re}(G(z)) > 0$.

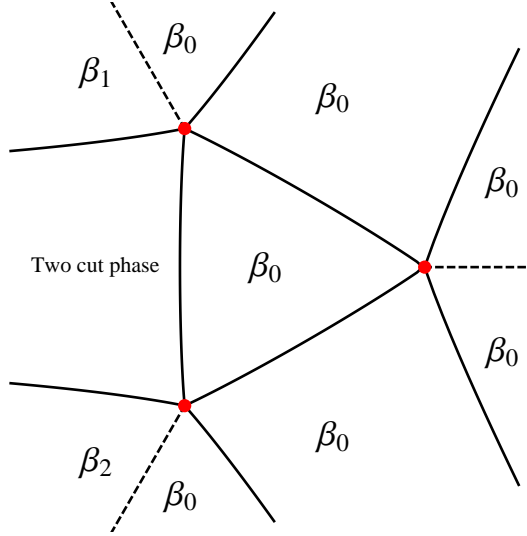


Figure 8: One cut phase (with the function β_k associated to the asymptotic zero density support), and two cut phase in the complex t -plane for the cubic model.

results in figure 8. We have labeled the regions corresponding to the one cut phase with the function $\beta_k(t)$ determining the asymptotic zero density support

4. The two-cut case

In the two-cut case we will denote a, b, c and d the endpoints of the cuts. Now we have

$$y(z) = \sqrt{(z-a)(z-b)(z-c)(z-d)}, \quad f(z) = -4z + b_0. \quad (46)$$

By substituting (2) and (46) into (19) and comparing the coefficients of z , z^2 and z^3 we obtain the equations

$$abc + abd + acd + bcd = 4, \quad (47)$$

$$ab + ac + bc + ad + bd + cd = -2t, \quad (48)$$

$$a + b + c + d = 0. \quad (49)$$

A new equation can be derived from (17), i.e., by imposing

$$\text{Re}(L_1) = \text{Re}(L_2). \quad (50)$$

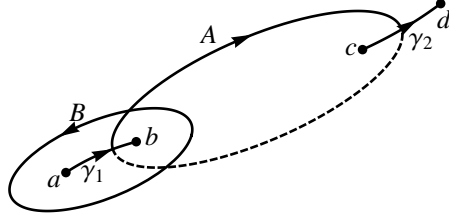


Figure 9: Homology basis.

In order to write this equation in terms of the endpoints we consider the hyperelliptic Riemann surface M associated to the spectral curve

$$y^2 = (z - a)(z - b)(z - c)(z - d). \quad (51)$$

The two branches $y_1(z) = -y_2(z) = y(z) \sim z^2 + \mathcal{O}(z)$ as $z \rightarrow \infty$ characterize M as a double-sheeted covering of the extended complex plane

$$M = M_1 \cup M_2, \quad M_i = \{Q = (y_i(z), z)\}, \quad (52)$$

and the meromorphic differential $y(z)dz$ in M . The homology basis $\{A, B\}$ of cycles in M is defined as shown in figure 9. The asymptotic condition (20) implies

$$y(z)dz = \begin{cases} \left(z^2 - t - \frac{2}{z} + \mathcal{O}(z^{-2}) \right) dz, & \text{as } Q \rightarrow \infty_1, \\ \left(-z^2 + t + \frac{2}{z} + \mathcal{O}(z^{-2}) \right) dz, & \text{as } Q \rightarrow \infty_2. \end{cases} \quad (53)$$

The periods of $y(z)dz$ with respect to the homology basis are given by

$$\begin{aligned} \oint_A y(z)dz &= \int_{z_1}^{z_2} y_1(z)dz + \int_{z_2}^{z_1} y_2(z)dz \\ &= 2[W(z_2) - g(z_{2+}) - g(z_{2-})] - 2[W(z_1) - g(z_{1+}) - g(z_{1-})] \\ &= 2(L_2 - L_1), \quad \text{with } z_1 \in \gamma_1, \quad z_2 \in \gamma_2 \quad \text{arbitrary,} \end{aligned} \quad (54)$$

and

$$\oint_B y(z)dz = -2 \int_a^b y(z_+)dz = -4\pi i \int_{\gamma_1} |dz|\rho(z), \quad (55)$$

i.e.

$$\operatorname{Re} \left(\oint_B y(z)dz \right) = 0. \quad (56)$$

From (54) it is clear that condition (50) can be rewritten as

$$\oint_A y(z)dz = 2ir, \quad r \in \mathbb{R}. \quad (57)$$

Let us now introduce the Abelian differential $d\psi$ defined by

$$d\psi = \frac{1}{2}(z^2 - t)dz + \frac{1}{2} \frac{[(z^2 - t - 2z^{-1})y(z)]_{\oplus} + C_1}{y(z)}dz, \quad (58)$$

whose only poles are at ∞_1 and ∞_2 , such that

$$d\psi = \begin{cases} (z^2 - t - \frac{1}{z} + \mathcal{O}(z^{-2})) dz, & Q \rightarrow \infty_1, \\ (\frac{1}{z} + \mathcal{O}(z^{-2})) dz, & Q \rightarrow \infty_2, \end{cases} \quad (59)$$

and with the coefficient C_1 uniquely determined by the normalization condition

$$\oint_A d\psi = \frac{1}{2} \int_a^b \frac{[(z^2 - t - 2z^{-1})y(z)]_{\oplus} + C_1}{y(z_+)}dz = 0. \quad (60)$$

Likewise, we introduce the first kind Abelian differential $d\varphi$ defined by

$$d\varphi = \frac{C_2}{y(z)}dz, \quad (61)$$

with the coefficient C_2 determined by the normalization condition

$$\oint_A d\varphi = 2C_2 \int_b^c \frac{1}{y(z_+)}dz = 1. \quad (62)$$

Also, for later reference, we define

$$\mathcal{A}_n = \int_b^c \frac{z^n}{y(z_+)}dz, \quad \mathcal{B}_n = \int_a^b \frac{z^n}{y(z_+)}dz, \quad (63)$$

and note that in particular

$$C_2 = \frac{1}{2\mathcal{A}_0}. \quad (64)$$

From (53) and (59) it is clear that

$$\frac{1}{2}(y(z) + z^2 - t)dz - d\psi \quad (65)$$

is a first kind Abelian differential in M . Consequently there exists $\lambda \in \mathbb{C}$ such that

$$\frac{1}{2}(y(z) + z^2 - t)dz - d\psi = \lambda d\varphi, \quad (66)$$

or

$$y(z)dz = -(z^2 - t)dz + 2d\psi + 2\lambda d\varphi. \quad (67)$$

Recalling (57), the A -period of (67) gives

$$\lambda = ir, \quad (68)$$

and the B -period is

$$\oint_B y(z)dz = 2 \oint_B d\psi + 2ir \oint_B d\varphi, \quad (69)$$

which together with (56) gives us with an expression for r

$$r = \frac{\operatorname{Re} \left(\oint_B d\psi \right)}{\operatorname{Im} \left(\oint_B d\varphi \right)}. \quad (70)$$

Thus, the remaining equation to determine the endpoints of the cuts is given by (57) together with (70). Taking into account that (47)–(49) imply

$$[(z^2 - t - 2z^{-1})y(z)]_{\oplus} = z^4 - 2tz^2 - 4z + C_0 \quad (71)$$

for a certain C_0 depending on the endpoints, we can also write

$$d\psi = \frac{1}{2}(z^2 - t)dz + \frac{1}{2} \frac{z^4 - 2tz^2 - 4z + C}{y(z)} dz, \quad C = C_0 + C_1, \quad (72)$$

and the normalization condition (60) gives

$$C = -\frac{\mathcal{A}_4 - 2t\mathcal{A}_2 - 4\mathcal{A}_1}{\mathcal{A}_0}. \quad (73)$$

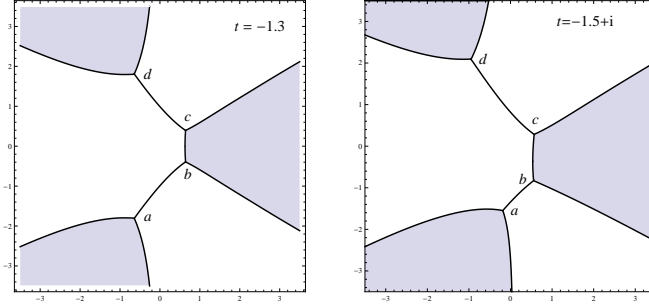


Figure 10: Two cut solutions for two choices of t in region 9.

Likewise, from (70), (63) and (64) we have that

$$r = \frac{\operatorname{Re}(\mathcal{B}_4 - 2t\mathcal{B}_2 - 4\mathcal{B}_1 + C\mathcal{B}_0)}{\operatorname{Im}(\mathcal{B}_0/\mathcal{A}_0)}. \quad (74)$$

Thus, the system (47)–(49) can be completed with the equation

$$\int_b^c y(z_+) dz = 2ir, \quad (75)$$

together with (73)–(74) to determine the endpoints.

It is clear that in general the system (47)–(49), (75) must be solved numerically. But even so, it would be very difficult to attempt a direct numerical solution without a well identified initial approximation. However, we can take advantage of our knowledge of the critical curves (44) and the corresponding explicit solutions for the one-cut endpoints given by (31), and proceed iteratively by small increments in t using as initial approximation at each step the results of the previous one. Once the endpoints a , b , c and d for a certain value of t have been calculated, the corresponding Stokes lines are also calculated numerically.

We plot in figure 10 the cuts and set of Stokes lines emerging from the endpoints for two choices of t in region 9. As in the one cut case we shade the region in which the condition $\operatorname{Re}(G(z)) > 0$ is satisfied. It is clear that the two cuts can be prolonged to an infinite curve homologue to Γ into the shaded sectors.

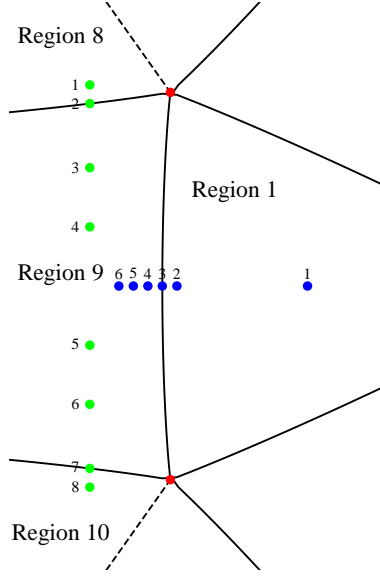


Figure 11: Phase transitions: the splitting of one cut (blue) and the birth and death of a cut at a distance (green)

5. Phase transitions in the cubic model

According to figures 3 and 8, phase transitions in the cubic model (processes where the number of cuts changes) can happen when the complex parameter t crosses the critical lines defining region 9. We present two examples:

- 1) First, we proceed along the negative t axis from region 1 beyond the critical value $t_{c1} \approx -1.00054$ into region 9 (blue t points in figure 11).
- 2) Second, we cross vertically from region 8 into region 9 beyond the critical value $t_{c2} \approx -1.5 + 1.54184$ and continue down to region 10 beyond the symmetric critical point $\overline{t_{c2}}$ (green t points in figure 11).

The splitting of a cut

In figure 12 we plot the set of Stokes lines emerging from the endpoints, and the regions $\text{Re}(G(z)) > 0$ for the first process corresponding to the blue points in figure 11. The process starts in region 1 with a one-cut solution. As t moves to the left the cut goes to the right, coming closer to the region

in the right in which $\text{Re}(G(z)) > 0$. In the critical value t_{c1} the cut forms a cusp as it arrives to the shaded regions, and then splits into two cuts. Thus, we have found a “splitting of a cut” in agreement with the theoretical result of [16].

Birth of a cut at a distance

In figure 13 we have crossed vertically from region 8 into region 9. In this case two infinite Stokes lines (*legs*) meet (second plot in figure 13). As we go down into region 9 a new cut is born from this crossing point. We have found a “birth of a cut at a distance”. The new cut increases while the initial cut decreases until it reduces to a point (in the seventh plot in figure 13) when we cross the critical curve and go to region 10, in which the cut disappears. We have found a symmetric “death of a cut at a distance”.

6. Asymptotic zero densities of orthogonal polynomials

As a check of the consistency of the results in the previous section with Theorem 1, in this section we will superimpose to the graphs of figures 12 and 13 the zeros of the corresponding polynomials $P_n(z)$ with degree $n = 36$.

First, we describe the algorithm for the computation of the polynomials. The definition (1) of the monic orthogonal polynomials can be rewritten as

$$\int_{\Gamma} e^{-nW(z)} P_k(z) P_j(z) dz = \delta_{kj} h_k, \quad j, k = 0, 1, \dots, n, \quad (76)$$

where the orthogonal polynomials satisfy the three term recursion relation

$$z P_k(z) = P_{k+1}(z) + s_k P_k(z) + r_k P_{k-1}(z), \quad k \geq 0, \quad (77)$$

with

$$r_0 = 0 \quad \text{and} \quad r_k = \frac{h_k}{h_{k-1}} \quad k \geq 1. \quad (78)$$

(Note that the previous equations implicitly assume that all the quantities involved are well defined, something a priori not obvious because the weight is complex valued.)

We start from the identity

$$0 = \int_{\Gamma} \frac{\partial}{\partial z} [e^{-nW(z)} P_k(z) P_j(z)] dz, \quad (79)$$

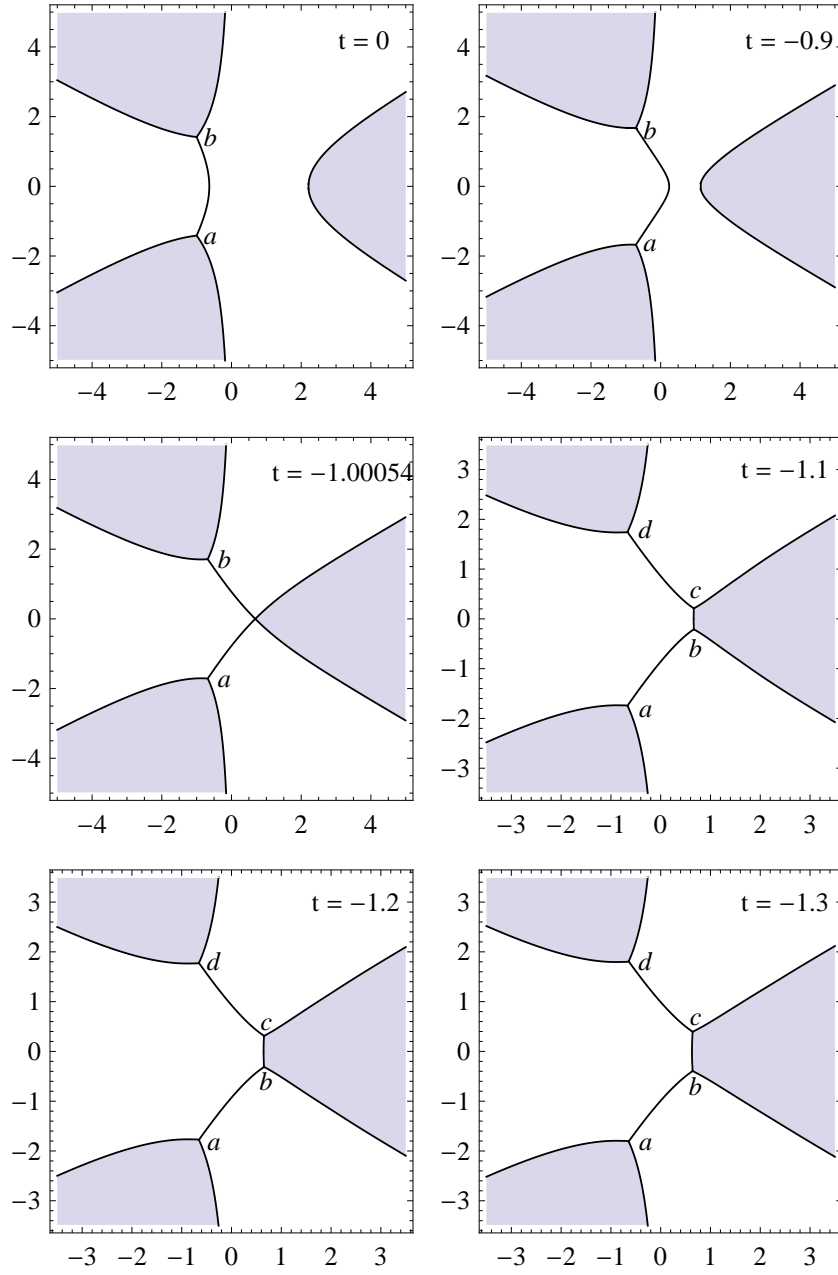


Figure 12: Splitting of a cut.

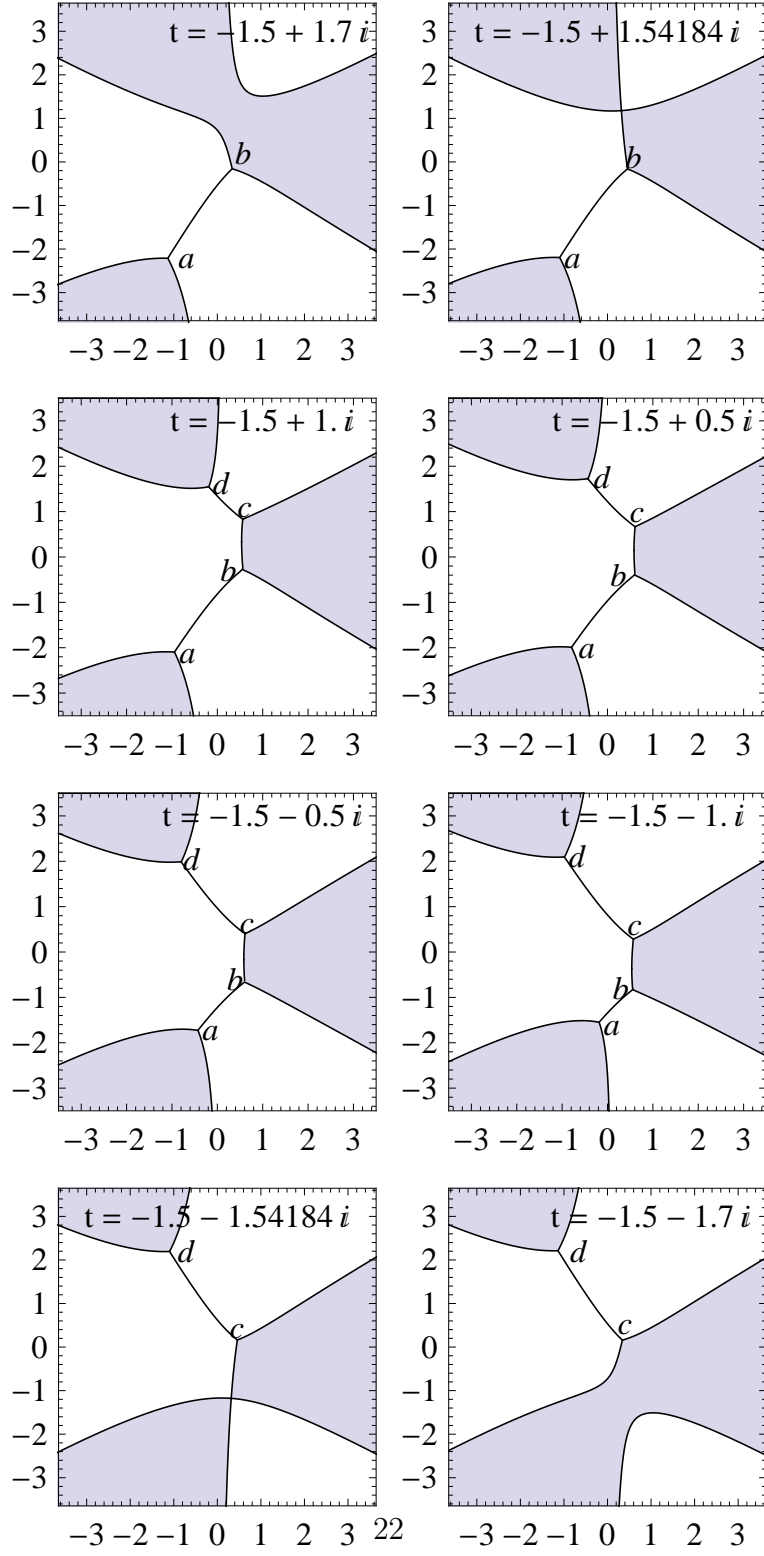


Figure 13: Birth and death of a cut at a distance .

written for the cubic model (2) as

$$0 = \int_{\Gamma} e^{-n\left(\frac{z^3}{3}-tz\right)} \left[-n(z^2-t)P_k(z)P_j(z) + P'_k(z)P_j(z) + P_k(z)P'_j(z)\right] dz. \quad (80)$$

Setting $j = k - 1$ in (80) and taking into account (77) it follows that

$$s_k + s_{k-1} - \frac{k}{nr_k} = 0. \quad (81)$$

Analogously, if we take $j = k$ in (80) and use (77) it is found that

$$r_{k+1} + r_k + s_k^2 - t = 0. \quad (82)$$

On the other hand, from (77)–(78) with $k = 0$ it is clear that

$$P_1(z) = z - s_0 \quad \text{and} \quad s_0 = \frac{\int_{\Gamma} e^{-n\left(\frac{z^3}{3}-tz\right)} z dz}{\int_{\Gamma} e^{-n\left(\frac{z^3}{3}-tz\right)} dz}. \quad (83)$$

Thus, we can compute $P_n(z)$ using the algorithm

$$\left. \begin{aligned} r_0 &= 0, \quad s_0 = \frac{\int_{\Gamma} e^{-n\left(\frac{z^3}{3}-tz\right)} z dz}{\int_{\Gamma} e^{-n\left(\frac{z^3}{3}-tz\right)} dz} \\ r_k &= t - r_{k-1} - s_{k-1}^2, \\ s_k &= \frac{k}{nr_k} - s_{k-1} \end{aligned} \right\} \quad k = 1, \dots, n-1 \quad (84)$$

$$P_n(z) = (z - s_{n-1})P_{n-1}(z) - r_{n-1}P_{n-2}(z).$$

In this way, for the computation of $P_n(z)$ we only need numerical approximation for two integrals. We take $n = 36$ and for the integrals in s_0 chose the path parametrized by

$$z(t) = \cosh\left(t + \frac{2\pi}{3}i\right), \quad (85)$$

We approximate the integral in the interval $t \in [-2, 2]$ and use a numerical adaptative quadrature implemented in *Mathematica*. Finally we superimposed in figures 14 and 15 the zeros of $P_{36}(z)$ for the values of t in figure 11 with our figures 12 and 13. In figure 14, which exemplify the splitting of a cut, as t decreases along the negative real axis (and due to the symmetry of the situation), the 36 zeros split evenly into the two sets of 18 zeros following closely the positions of the cuts that correspond to the limit $n \rightarrow \infty$. In figure 15, which exemplifies the birth and death of a cut at a distance, what we find numerically as the value of t descends vertically from $t = -1.5 + 1.5i$ to $t = -1.5 - 1.5i$ is that all the 36 zeros lie initially on the lower cut ab , and start travelling upwards one by one, thus populating the upper cut cd and depopulating the lower ab . This behavior is particularly clear in the third and fourth graph (corresponding to $t = -1.5 + i$ and $t = -1.5 + 0.5i$), in which the sixth and 12th zero are “arriving” at the upper cut. Also in the symmetric fifth and sixth graph (corresponding to $t = -1.5 - 0.5i$ and $t = -1.5 - i$), we can see the 25th and 31th zeros “leaving” the lower cut.

Acknowledgments

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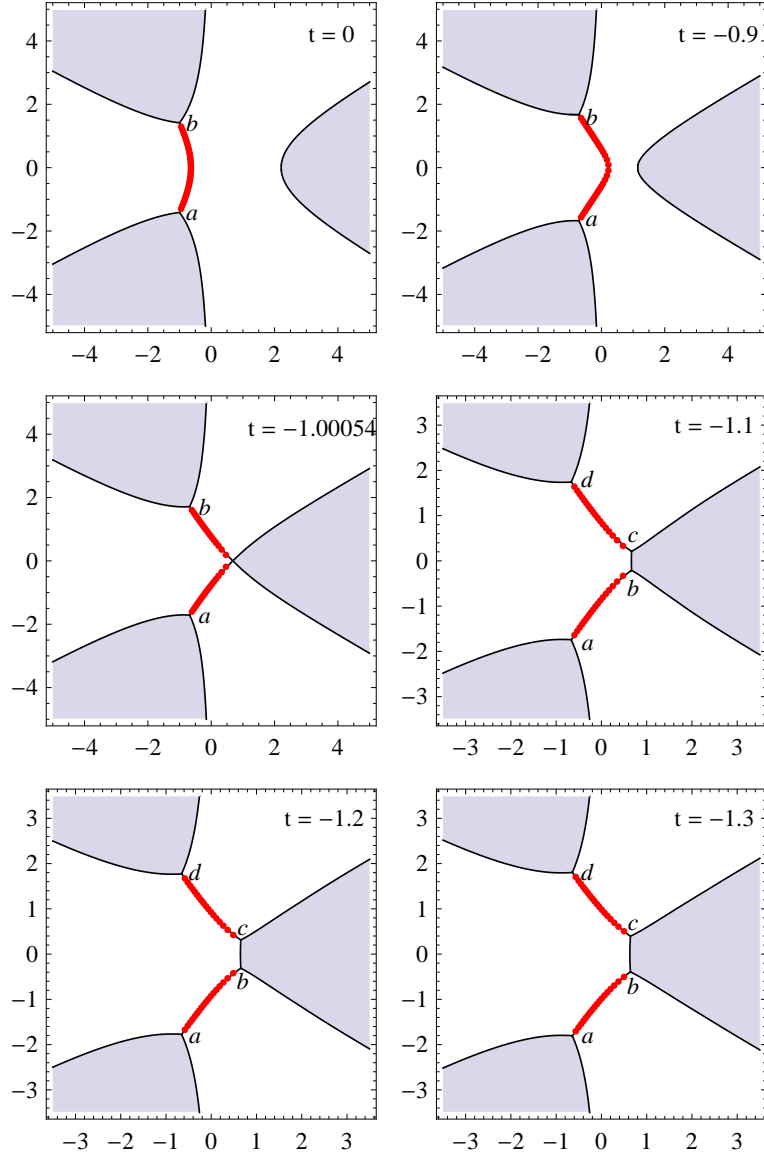


Figure 14: Zeros of $P_{36}(z)$ superimposed to the splitting of a cut in figure 12.

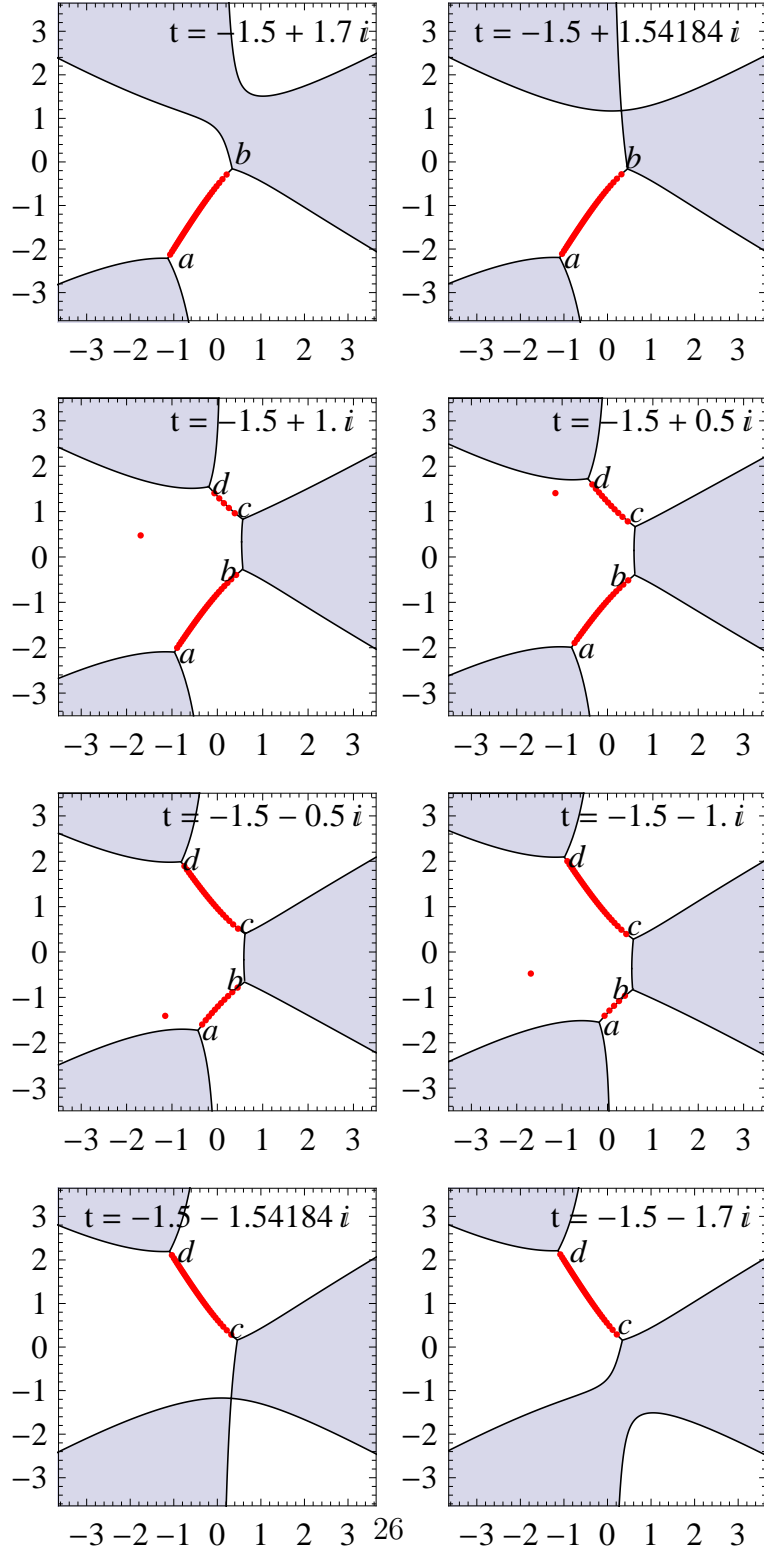


Figure 15: Zeros of $P_{36}(z)$ superimposed to the birth and death of a cut at a distance in figure 13.

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